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ON THE VIBRATION OF THIN CYLINDRICAL SHELLS UNDER INTERNAL PRESSURE

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Aeromechanics Section

Approved ______________________
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SUMMARY

The frequency spectra and vibration modes of thin-walled cylinders subjected to internal pressure are considered. The cylinders are supported in such a manner that the ends remain circular.

The basic mathematical analysis has been given by Reissner. In the present note the interesting physical features of the vibration of thin cylindrical shells are discussed. Further theoretical investigations are made and are applied to the case in which the number of circumferential waves is relatively small, thus complementing Reissner's theory which applies only when this number is sufficiently large.

It is shown that for thin cylinders the internal pressure has a very significant effect on the natural vibration characteristics. For these cylinders, particularly those having smaller length-to-diameter ratios, the mode associated with the lowest frequency is in general not the simplest mode. In fact, the frequency spectrum may be so arranged that the frequency decreases with an increase in the number of circumferential nodes. The exact number of circumferential nodes which occur in the mode associated with the lowest frequency depends on the ratios h/a, L/a, and the internal pressure, where h is the wall thickness, a is the cylinder radius, and L is the axial half-wave length. When the internal pressure is small, the number of circumferential nodes at the lowest frequency decreases rapidly with increasing internal pressure; and the "fundamental" frequency — the lowest frequency at each internal pressure — increases rapidly with increasing internal pressure. At higher values of internal pressure the frequency spectrum tends to be arranged in the regular manner: the frequency increases with the increasing number of circumferential nodes; and the lowest frequency rises with the internal pressure, but at a slower rate.

The spacing of the successive frequencies of the natural vibration of a cylinder is irregular and often very close. A knowledge of the entire spectrum is therefore necessary in order to avoid possible resonance in engineering applications.

NOTATIONS

a	radius of cylinder.
a_1, a_2, b_1, b_2 etc.	coefficients in K_o', K_1', K_2'.
A, B, C	maximum amplitude of component vibrations.
e_x, e_y, e_z	strain components
E	Young's modulus
$f = \omega/2\pi$	frequency, cycles per sec.
h	thickness of wall
K_o, K_1, K_2, K_o' etc.	coefficients in frequency equation.
L	length of cylinder
m	number of axial half-waves
M_x, M_φ, $M_{x\varphi}$	Moments in shell, per unit length.
n	number of circumferential waves (number of nodes = 2n)
N_x, N_φ, $N_{x\varphi}$, Q_x, Q_y	stress resultants in shell, force per unit length.
$\bar{N}_x$, $\bar{N}_\varphi$	membrane stresses due to internal pressure.

$$\bar{n}_\varphi = \frac{\bar{N}_\varphi}{E\,h}$$

$$n_\varphi = (1-\nu^2)\,\bar{n}_\varphi$$

$$\bar{n}_x = \frac{\bar{N}_x}{E\,h}$$

$$n_x = (1-\nu^2)\,\bar{n}_x$$

t	time.
u, v, w	components of displacement of a point on the mid-surface of the shell.
x, $y = a\varphi$, z	coordinates in axial, circumferential, and radial directions

$$\alpha = h/a$$

$$\beta = \frac{h^2}{12a^2}$$

ϵ_x, ϵ_φ, $\gamma_{x\varphi}$ strain in the mid-surface of the shell.

$\varkappa = \frac{\rho \omega^2 a^2}{E}$ frequency factor

$\Delta = (1-\nu^2)\varkappa$

$\lambda = \frac{\pi m a}{L}$ axial wave length factor.

ν Poisson's ratio

ρ density of shell material

$\sigma_x, \sigma_y, \tau_{xy}$ stress components

φ angular coordinate

$\chi_x, \chi_\varphi, \chi_{x\varphi}$ change of curvature of the mid-surface of the shell.

ω circular frequency = $2\pi f$.

THE VIBRATION OF THIN CYLINDRICAL SHELLS UNDER INTERNAL PRESSURE

by

Y. C. FUNG

INTRODUCTION

The physical problem of vibration of thin cylindrical shells is not without elements of surprise. Thus it has been found that the natural frequencies are arranged in an order which had little relation to the complexity of the nodal pattern. But such phenomenon has been explained by Arnold and Warburton (Ref. 1, 1949), whose accurate theoretical analysis and experimental investigations demonstrate beyond doubt the adequacy of Timoshenko's equations (or Love's "first approximation") in describing the vibration of unpressurized cylinders. On the other hand, the effect of internal pressure on the free vibration of thin cylinders does not appear to be generally appreciated. In Ref. 1, it is stated that "the results (of analysis of unpressurized cylinders) are also true for a cylinder subjected to uniform stress; thus internal fluid pressure in a container will not affect the frequency". The fact is, however, that the internal pressure has a grave effect on the frequency. For instance, one may easily construct realistic examples in which an internal pressure that induces a hoop stress of order 0.2 per cent of the ultimate strength of the cylinder causes an increase of 10 per cent in the lowest frequency over that of an unpressurized cylinder (we built and tested such a cylinder). The importance of internal pressure on the vibration of pressure vessels is quite evident.

Of the great variety of ways in which a thin cylindrical shell may vibrate, only the flexural vibration of the wall of the cylinder will be considered. Both bending and stretching of the shell are involved. The

cylinders are assumed to be "freely supported" in such a manner that the ends remain circular, and that no restraint on the axial or tangential displacement is imposed at the ends.

The large apparent effect of the internal pressure on the lowest natural frequency can be explained by the same fact as noted by Arnold and Warburton in the case of unpressurized cylinders: that for cylinders with very thin walls the mode of the lowest frequency is in general not the simplest one. For a given axial wave length, the exact number of circumferential nodes that occur in the mode of the lowest frequency depends on the bending stiffness of the cylinder wall and the internal pressure. This number is rather sensitive to the internal pressure when the cylinder is very thin and when the pressure is small. Thus a relatively small change in internal pressure may cause considerable change in the mode shape at the lowest frequency. When the internal pressure is large, the fundamental frequency varies approximately as the square root of the pressure. Since at very large internal pressure a thin cylinder derives its stiffness mostly from the hoop tension, the last mentioned result seems natural in comparison with the corresponding cases of vibration of a stretched string or a stretched membrane.

There is a basic difference in the frequency spectrum of a cylinder as compared to those of some other more familiar examples, such as strings, membranes, beams, and plates. the difference in the spacing of successive frequencies. For the transverse vibration of a string the successive frequencies are spaced like successive integers n. For the transverse vibration of a simply supported beam the frequencies are spaced like n^2, (n an integer). For a rectangular membrance the transverse vibration frequencies are

$$f_{mn} \sim \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad \begin{array}{l}(m,\ n \text{ integers})\\ (a, b \text{ dimensions of membrane})\end{array}$$

For a simply supported rectangular plate, they are

$$f_{mn} \sim \frac{m^2}{a^2} + \frac{n^2}{b^2} \qquad (m,\ n,\ \text{integers})$$

Beams, membranes, and plates of other boundary conditions and shapes have less regular spacing than those mentioned above, but the general character is similar. In constrast to this, an examination of the frequency spectra of a cylinder will show that the successive frequencies rarely follow any simple rule. They do not follow an arithmatic progression as in strings or membranes, or as n^2 as in beams or plates. Often several successive frequencies are very closely grouped together.

The last point has an important bearing on the engineering applications of the theory. Whereas knowledge about the lowest frequencies is often sufficient in ordinary problems, for cylinder it is quite possible that a more complete knowledge about the entire frequency spectrum is essential in order that resonance can be avoided.

HISTORY OF THE PROBLEM

(a) Unpressurized Cylinders

Previous investigations on the vibration of thin-walled cylinders seem to be limited mostly to the unpressurized case. The free vibrations of thin cylinders in the case of negligible bending stiffness has been considered by Rayleigh (Ref. 2, 1894). The solution is relatively simple. Inclusion of the bending stiffness of the walls of the shells makes the problem much more complicated. Rayleigh (1894) derived an expression for the frequencies of thin cylinders in which the motion of all cross-sections was identical. This corresponds to the fundamental axial form for a free-ended cylinder. The general equations of flexural vibration of the walls of cylinders were later investigated by Love (Ref. 3, 1927), but no frequency equations for any specific end conditions were given. Flügge (Ref. 4, 1934) first gave the frequency equation for a cylinder with freely supported ends; but extensive numerical results were first published by Arnold and Warburton (Ref. 1, 1949, 1953). For freely supported ends, Arnold and Warburton derived frequency equations based on strain energy relations due to Timoshenko (Ref. 5, 1940) and were able to verify the experimental results with considerable accuracy. The fact that for short cylinders with very thin walls the natural frequency may actually decrease as the number of circumferential nodes increases was shown theoretically to be due to the proportion of strain energy contributed respectively by bending and stretching; the latter was sometimes predominant for the simpler nodal patterns. In their 1953 paper, the effect of various end conditions was discussed exhaustively both from the theoretical and experimental points of view.

Tables for frequencies and modes of free vibration of unpressurized cylinders of infinite length were published also by Baron and Bleich (Ref. 6, 1954), who computed first the frequencies of the modes of a selected wave length using the membrane theory, and then estimated the corrections due to bending stiffness of the walls on the basis of Rayleigh's principle.

(b) Pressurized Cylinders

The vibration of pressurized cylinder has been discussed by Stern (Ref. 7, 1954) under the hypotheses that (1) the skin vibrates normal to the

static position and that (2) the stresses in the vibration mode are effectively equal to the stresses generated by the internal pressure. Later, Serbin (Ref. 8, 1955) solved the problem for the lowest frequencies for "nearly-inextensional" vibration modes by Rayleigh's method.

Recently, Reissner (Ref. 9) showed that a great simplification in the in the analysis of shell vibrations can be achieved if the tangential inertia forces can be neglected. It is then necessary to consider only one component of displacement — the transverse, or radial, component. The resulting equations are precisely Marguerre's equations for slightly curved plates (a generalization of Von Kármán's equations for flat plates). Reissner shows that the simplified equations — called "shallow shell" theory — will give an accurate description of the transverse vibration of cylinders provided that the number of circumferential waves, n, is sufficiently large. In Ref. 10, Reissner gives the following frequency expression for transverse vibrations of small amplitude:

$$\rho h \omega^2 = \frac{Eh}{a^2} \frac{(\pi/L)^4}{\left[\left(\frac{n}{a}\right)^2 + \left(\frac{\pi}{L}\right)^2\right]^2} + \frac{Eh^3}{12(1-\nu^2)}\left[\left(\frac{n}{a}\right)^2 + \left(\frac{\pi}{L}\right)^2\right]^2 + \bar{N}_x\left(\frac{\pi}{L}\right)^2 + \bar{N}_\varphi\left(\frac{n}{a}\right)^2, \tag{1}$$

where ρ is the density, h is the wall thickness, ω is the circular frequency, E is the Young's modulus, ν is the Poisson's ratio, a is the radius, L is the axial half-wave length, $\bar{N}_x$ is the axial stress resultant per unit length, and $\bar{N}_\varphi$ is the circumferential stress resultant per unit length. This expression corresponds to a deflection form

$$w = C \sin \omega t \sin \frac{\pi x}{L} \cos \frac{ny}{a} . \tag{2}$$

The first term on the right hand side of Eq. (1) gives the influence of stretching of the shell, the second term, of bending; and the last two terms, of internal pressure.

CALCULATIONS FOR SMALL NUMBER OF CIRCUMFERENTIAL WAVES

Reissner's theory applies to vibration modes with sufficiently large number of circumferential waves. In order to complete the theoretical investigation it is only necessary to investigate the case in which the number of circumferential waves, n, is relatively small. The mathematical analysis is given in Appendix A. The coordinate system is shown in Fig. 1. The axial, circumferential, and radial displacements are assumed to be of the form:

$$\begin{aligned} u &= A \cos \frac{m\pi x}{L} \cos n\varphi \cos \omega t \\ v &= B \sin \frac{m\pi x}{L} \sin n\varphi \cos \omega t \\ w &= C \sin \frac{m\pi x}{L} \cos n\varphi \cos \omega t \end{aligned} \qquad (3)$$

where m is the number of axial half-waves, L is the length of the cylinder, and A, B, C are constants. Following Arnold and Warburton, the end conditions corresponding to the above are called "freely supported". Displacements given by (3) satisfy the differantial equations of motion. It is clear that whereas w and v vanish at all times at the ends, u does not, corresponding to the case in which no axial constraint is imposed. Furthermore, the radial movement w vanishes at $2n$ equally spaced lines along the circumference, but these lines are not lines of absolute rest, because the tangential motion v does not vanish identically there. For convenience, however, lines on which w is stationary will be called nodal lines, even though tangential motion exists. A pair of integers m, n then specifies a particular nodal pattern as shown in Fig. 2.

The frequency equation so obtained are of the form

$$\Delta^3 - K_2' \Delta^2 + K_1' \Delta - K_0' = 0 \qquad (4)$$

where Δ is a parameter proportional to frequency squared, and the K's are functions of the axial and circumferential wave number, internal pressure, and physical dimensions of the cylinder. Under various additional simplifying assumptions the frequency equation (4) can be simplified. The simplification introduced by Reissner and Stern are discussed in Appendix B. Eq. (4), of course, applies to all values of m and n, not only when they are small.

In order to observe the manner in which the frequency varies with the cylinder dimensions or nodal pattern, calculation for specific cases was undertaken. Four non-dimensional parameters introduced in Ref. 1 are used*, namely

$$\sqrt{\kappa} = \text{frequency} \times \sqrt{\frac{4\pi^2 a^2 \rho}{E}} \,, \qquad \kappa = (1-\nu^2)\Delta$$

$$\lambda = \frac{\text{mean circumference}}{\text{axial wave-length}} = \frac{2\pi a}{2L/m} = \frac{m\pi a}{L} \tag{5}$$

$$n = \text{number of circumferential waves (number of nodes} = 2n).$$

$$\alpha = \frac{\text{thickness}}{\text{mean radius}} = \frac{h}{a}$$

The effect of internal pressure is expressed by the non-dimensional parameters

$$\bar{n}_\varphi = \frac{\bar{N}_\varphi}{Eh}$$

$$\bar{n}_x = \frac{\bar{N}_x}{Eh} \tag{6}$$

where $\bar{N}_\varphi$, $\bar{N}_x$ are the membrane stresses caused by the internal pressure p. For example, in a boiler,

$$\bar{N}_\varphi = pa \,, \qquad \bar{N}_x = pa/2$$

We shall now give a few examples:

Example 1 For $\nu = 0.3$, $\lambda = 1$, $a/h = 1750$, $\bar{n}_\varphi = \bar{n}_x = 0$.
Then the roots Δ are

$n=1$	$n=2$	$n=3$	$n=4$
0.124	0.0285	0.00857	0.00407
0.994	1.93	3.611	6.02
2.582	5.79	10.88	17.93

* κ differs from Δ of Ref. 1 by a factor $(1-\nu^2)$. Both Δ and κ will be used in the present report.

For $\nu = 0.3$, $\lambda = 1$, $a/h = 100$, $\bar{n}_\varphi = \bar{n}_x = 0$, then the roots Δ are

n = 1	n = 2	n = 3	n = 4
0.124	0.0286	0.00870	0.00493
0.994	1.93	3.611	6.022
2.582	5.79	10.88	17.93

These are unpressurized cylinders.

Example 2 If $\nu = 0.3$, $\lambda = 1$, $a/h = 1750$, $\beta = \frac{1}{12}\left(\frac{1}{1750}\right)^2$, $\bar{n}_\varphi = 10^{-4}$, $\bar{n}_x = \frac{1}{2}\bar{n}_\varphi$,

then the roots Δ are

n = 1	n = 2	n = 3	n = 4
0.124	0.0285	0.00857	0.00407
0.994	1.93	3.612	6.02
2.582	5.79	10.88	17.93

Example 3 Same constants as above, except $\bar{n}_\varphi = 5 \times 10^{-3}$, $\bar{n}_x = \bar{n}_\varphi/2$.
The roots Δ are

n = 1	n = 2	n = 3	n = 4
0.122	0.0313	0.0310	0.0585
1.008	1.96	3.64	6.05
2.581	5.79	10.88	17.92

Example 4 Same data as before, except $\frac{a}{h} = 100$, so that $\beta = \frac{10^{-4}}{12}$.
The roots Δ are, for $\bar{n}_\varphi = 2\bar{n}_x = 10^{-4}$,

n = 1	n = 2	n = 3	n = 4
0.124	0.0286	0.00915	0.00605
0.994	1.93	3.612	6.022
2.582	5.79	10.88	17.93

When $\bar{n}_\varphi = 2\bar{n}_x = 5 \times 10^{-3}$, we have

n = 1	n = 2	n = 3	n = 4
0.128	0.0314	0.0315	
0.999	1.96	3.642	
2.584	5.79	10.88	

We note that $n = 1$ implies a bending vibration. The end sections have considerable longitudinal motion. The calculation, however, is valid only if there is no concentrated masses to be moved at the ends. The very large stretching energy involved in this mode explains the very high frequency. In practical applications there are usually heavy masses at the ends of the cylinder and the frequency must be computed accordingly.

Similar remarks apply to other cases in which n is small. However, if the masses supported at the ends were rigidly connected, their effect on frequency for $n \geqslant 2$ should be small. In other words, $n = 1$ really is a case of different catagory than the cases $n \geqslant 2$.

Comparing the results of the preceeding examples, we see that the variation of Δ with $\bar{n}_\varphi$, $\bar{n}_x$ is small for small n. The smallest root Δ begin to vary significantly with $\bar{n}_\varphi$, $\bar{n}_x$ when $n > 3$, but the two larger roots do not vary much with $\bar{n}_\varphi$.

The non-dependance of the larger roots on the internal pressure is simply explained by the fact that these roots correspond to essentially tangential motions, and that the hoop stress has little stiffening effect for tangential motions.

It is further noted that the two larger roots are considerably in excess of the smallest roots in a very wide range of $\bar{n}_\varphi$. Hence unless one is interested in a frequency range which includes frequencies 20 or 30 times larger than the minimum frequency, one need not consider the predominantly tangential vibration modes.

Figs. 3 & 4 show the frequency as a function of λ and n for very thin-walled unpressurized cylinder (these curves complements those of Ref. 1). Curves in Figs 3a and 4a are drawn for $\alpha = 1/1750$, Fig. 3b for $\alpha = 2/1750$, and Fig. 4b for $\alpha = 3/1750$, $\nu = 0.3$. The influence of α on these curves is small at such small value of α, as can be seen by a comparison of Figures 3a and 3b. For small values of λ, however, the frequency is strongly influenced by α. The details of the variations of frequency with λ when λ is small are shown in enlarged scales in Figs. 4a and 4b.

Figs. 3a and 4a also give a graphical comparison of Reissner's solution with the more laborious exact solution. The numerical accuracy of Reissner's solution deteriorates a little as n decreases. But it is seen that Reissner's equation gives a fairly accurate representation for all values of n.

Since in Reissner's approximation certain inertia forces are neglected, it is expected that the derived frequency will be higher than the exact value. This is readily seen to be the case.

THE MODE ASSOCIATED WITH THE LOWEST FREQUENCY

It is of interest to observe some simple deductions with regard to the lowest frequency and mode.

First let Eq. (1) be rewritten in the non-dimensional form

$$\mathcal{K} = \frac{\lambda^4}{(n^2+\lambda^2)^2} + \frac{\alpha^2}{12(1-\nu^2)}(n^2+\lambda^2)^2 + \bar{n}_x \lambda^2 + \bar{n}_\phi n^2. \tag{7}$$

where $\alpha = h/a$. For given values of λ, α, $\bar{n}_x$ and $\bar{n}_\phi$, the frequency facto[r] $\mathcal{K}$ reaches a minimum when

$$\frac{\partial \mathcal{K}}{\partial n} = 2n\left[-\frac{2\lambda^4}{(n^2+\lambda^2)^3} + \frac{\alpha^2}{6(1-\nu^2)}(n^2+\lambda^2) + \bar{n}_\phi\right] = 0 \tag{8}$$

If $\bar{n}_\phi = 0$, this gives

$$n = \sqrt{\left[12(1-\nu^2)\right]^{1/4}\frac{\lambda}{\alpha^{1/2}} - \lambda^2} \tag{9}$$

If $\lambda \ll \sqrt{\frac{a}{h}}$, then

$$n \sim \sqrt{2\lambda}\left(\frac{a}{h}\right)^{1/4} \tag{10}$$

and

$$\mathcal{K}_{min} = \frac{\lambda^2 \alpha}{\sqrt{3(1-\nu^2)}} \tag{11}$$

Of course, n must be an integer and the above minimum value may not necessarily be reached.

If $\bar{n}_\varphi \gg \lambda \alpha^{3/2}$, i.e. $\frac{p}{E} \gg \lambda \left(\frac{h}{a}\right)^{5/2}$, then

$$n^2 \doteq \frac{\sqrt[3]{2}\ \lambda^{4/3}}{\bar{n}_\varphi^{1/3}} - \lambda^2 \tag{12}$$

When the internal pressure does not vanish the variation of the number of circumferential waves at the lowest frequency is best illustrated by an example. Fig. 5 shows a special case in which the axial wave length parameter $\lambda = 1$, the bending stiffness parameter $\beta = 10^{-8}$, and $\bar{n}_\varphi = 2\bar{n}_x$. It is seen that n, the circumferential wave number corresponding to the lowest frequency, varies with the internal pressure parameter $\bar{n}_\varphi$: at first very rapidly, then slowly, as shown in the following table and Fig. 6.

n No. of circumferential waves at lowest freq.	$\bar{n}_\varphi \times 10^5$ Range of Internal Pressure Parameter $\bar{n}_\varphi \times 10^5$
10	0 — 0.1
9	0.1 — 0.38
8	0.38 — 0.97
7	0.97 — 2.5
6	2.5 — 7.0
5	7.0 — 22
4	22 — 94
3	94 — 610
2	610 — 7000
1	> 7000

The above table is based on Eq. (1). The range of $\bar{n}_\varphi$ for $n \leq 3$ is probably quite inaccurate.

To give some physical feeling about this example, it may be said that $\lambda = 1$ corresponds to a cylinder whose length/radius ratio is 3 or integral multiples of 3, and that $\beta = 10^{-8}$ corresponds to a radius/thickness ratio of order 3000. If $E = 10^7$, then $\bar{n}_\varphi = 10^{-5}$ when the internal pressure is only 0.03 psi.

It is clear that the larger the number of circumferential waves, n, the faster is the rate of increase of frequency χ with the internal pressure. The rapid increase in the lowest frequency when $\bar{n}_\varphi$ is small is caused by the fact that n is fairly large at the lowest frequency if the cylinder is short and if the wall is very thin.

THE FREQUENCY SPECTRUM

Since each pair of integers (m, n) determines a vibration mode, the frequency spectrum of a cylinder is obtained by computing χ for all values of (m, n). Now each selection of m determines a λ. Hence for a given m, the frequencies corresponding to all n are obtained by the intersections of the curves(such as those in Fig. 5) with vertical lines representing specific values of $\bar{n}_\varphi$. From Fig. 5 it is clear that the spacing of the spectrum is rather irregular. On many occasions two modes have the same frequency.

If the exact frequency determinant as given in the Appendix were used, there will be three frequencies corresponding to each flexural deflection pattern (w). One of these frequencies is predominantly flexural, and is approximated by Reissner's formula. The other two are associated with significant tangential motions (u, v), and have higher frequencies. These are of course lost in Reissner's approximation.

CONCLUSIONS

In the above, those physical features of cylinder vibration that are important in engineering applications are discussed. It seems that the classical shell theory is sufficient to explain the features of a vibrating cylinder under internal pressure. Reissner's simplified treatment is adequate in giving the lowest frequencies and modes for very thin cylinders. The complete frequency equation is given in the appendix of this article.

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APPENDIX A

DERIVATION OF FREQUENCY EQUATION FOR THIN CYLINDERS WITH FREELY SUPPORTED ENDS

The basic assumptions of Love's first approximation will be used. See Love, Ref. 3, Chapter 24; Timoshenko, Ref. 5, Chapter 11, and Naghdi and Berry, Ref. 11. To account for the effect of internal pressure, the interaction of the membrane stresses and the change of curvatures must be included in the equations of equilibrium. The basic equations can be obtained by a trivial generalization of the equations given in Timoshenko's book. Since the results did not seem to have been recorded in the literature, some details will be given below.

Let the curvilinear coordinate system be chosen as shown in Fig. 1; the x-axis is directed along the generator of the cylinder, $y = a\phi$ is measured clockwise in the circumferential direction, and the z-axis is directed inward along the positive normal to the middle surface of the shell. Let u, v, w be the components of the displacement of a point on the mid-surface of the shell in the x, y, z directions respectively.
The stress resultants and couples are defined by

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz , \qquad M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz , \qquad Q_x = \int_{-h/2}^{h/2} \tau_{xz} \, dz \tag{A.1}$$

The equations of motion are obtained by adding inertia force terms $-\rho h \, \partial^2 u/\partial t^2$, $-\rho h \, \partial^2 v/\partial t^2$, $-\rho h \, \partial^2 w/\partial t^2$, multiplied by the radius a, to the left hand side of Eqs. (252), p. 438, of Ref. 5. The resulting equations are non-linear. For the vibration problem, it is justifiable to linearize by considering motions of infinitesimal amplitude. Let $\overline{N}_x, \overline{N}_\varphi, \overline{M}_x, \overline{M}_\varphi$ etc. be the stress resultants and couples in the cylinder induced by the internal pressure. For simplicity, we shall assume that

$$\begin{gathered} \overline{N}_x = \text{const.}, \qquad \overline{N}_\varphi = \text{const.} \\ \overline{N}_{x\varphi} = \overline{Q}_x = \overline{Q}_\varphi = \overline{M}_x = \overline{M}_\varphi = \overline{M}_{x\varphi} = 0 \end{gathered} \tag{A.2}$$

In other words, the effect of any possible bending caused by the end conditions will be neglected in the vibration problem. Let u, v, w, N_x,

N_φ, M_x, etc. be the vibratory displacements and vibratory stress resultants and couples i.e. the variations from the uniform stress status induced by the internal pressure. u, v, w, N_x, N_φ, etc. are assumed to be of infinitesimal amplitude. Neglecting small quantitiies of the second or higher order, we obtain the equations of motion:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{\varphi x}}{a\partial\varphi} - \frac{\bar{N}_\varphi}{a}\left(\frac{\partial^2 v}{\partial x\partial\varphi} - \frac{\partial w}{\partial x}\right) - \rho h\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial N_{x\varphi}}{\partial x} + \frac{\partial N_\varphi}{a\partial\varphi} - \frac{Q_\varphi}{a} + \bar{N}_x\frac{\partial^2 v}{\partial x^2} - \rho h\frac{\partial^2 v}{\partial t^2} = 0 \qquad (A.3)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\varphi}{a\partial\varphi} + \frac{N_\varphi}{a} + \bar{N}_x\frac{\partial^2 w}{\partial x^2} + \frac{\bar{N}_\varphi}{a}\left(\frac{\partial v}{a\partial\varphi} + \frac{\partial^2 w}{a\partial\varphi^2}\right) - \rho h\frac{\partial^2 w}{\partial t^2} = 0$$

and

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{\varphi x}}{a\partial\varphi} - Q_x = 0$$

$$-\frac{\partial M_{x\varphi}}{\partial x} + \frac{\partial M_\varphi}{a\partial\varphi} - Q_\varphi = 0 \qquad (A.4)$$

Eliminating the shearing forces Q_x, Q_φ, and substituting the stress-strain relations, we obtain the basic equations*:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1+\nu}{2a}\frac{\partial^2 v}{\partial x\partial\varphi} - \frac{\nu}{a}\frac{\partial w}{\partial x} - \frac{\bar{N}_\varphi(1-\nu^2)}{Eha}\left(\frac{\partial^2 v}{\partial x\partial\varphi} - \frac{\partial w}{\partial x}\right) - \frac{1-\nu^2}{Eh}\rho h\frac{\partial^2 u}{\partial t^2} = 0,$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{a\partial x\partial\varphi} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2} - \frac{1}{a^2}\frac{\partial w}{\partial\varphi} \qquad (A.5)$$

$$+\frac{h^2}{12a^2}\left(\frac{\partial^3 w}{\partial x^2\partial\varphi} + \frac{\partial^3 w}{a^2\partial\varphi^3}\right) + \frac{h^2}{12a^2}\left[(1-\nu)\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{a^2\partial\varphi^2}\right] + \frac{1-\nu^2}{Eh}\left[\bar{N}_x\frac{\partial^2 v}{\partial x^2} - \rho h\frac{\partial^2 v}{\partial t^2}\right] = 0,$$

$$\nu\frac{\partial u}{\partial x} + \frac{\partial v}{a\partial\varphi} - \frac{w}{a} - \frac{ah^2}{12}\nabla^4 w - \frac{h^2}{12}\left(\frac{2-\nu}{a}\frac{\partial^3 v}{\partial x^2\partial\varphi} + \frac{\partial^3 v}{a^3\partial\varphi^3}\right)$$

$$+\frac{a(1-\nu^2)}{Eh}\left[\bar{N}_x\frac{\partial^2 w}{\partial x^2} + \frac{\bar{N}_\varphi}{a}\left(\frac{\partial v}{a\partial\varphi} + \frac{\partial^2 w}{a\partial\varphi^2}\right) - \rho h\frac{\partial^2 w}{\partial t^2}\right] = 0$$

---

*Stern's equation, Ref. 7, consists of neglecting all other terms in Eqs. (A.5) except the terms in the square bracket in the last equation of (A.5); the quantity $\partial v/a\partial\varphi$ being replaced by w/a under the assumption of zero circumferential strain. Reissner's approximation consists in neglecting the terms $\partial^2 u/\partial t^2$, $\partial^2 v/\partial t^2$ and eliminating u, v by means of a stress function.

where ∇^4 denotes the operator

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{a^2 \partial \varphi^2} \right)^2$$

This set of equations admit solutions in the form

$$\begin{aligned} u &= A \cos \frac{m\pi x}{L} \cos n\varphi \cos \omega t \\ v &= B \sin(m\pi x/L) \sin n\varphi \cos \omega t \\ w &= C \sin(m\pi x/L) \cos n\varphi \cos \omega t \end{aligned} \tag{A.6}$$

where A, B, C, are constants, and m, n are integers. Substituting (A.6) into (A.5), and writing

$$\begin{aligned} \Delta &= \frac{\rho(1-\nu^2)\omega^2 a^2}{E} = (1-\nu^2)\mathcal{H} \\ n_\varphi &= \bar{N}_\varphi (1-\nu^2)/Eh = (1-\nu^2)\,\bar{n}_\varphi \\ n_x &= \bar{N}_x (1-\nu^2)/Eh = (1-\nu^2)\,\bar{n}_x \\ \lambda &= m\pi a/L \\ \beta &= h^2/12a^2 \end{aligned} \tag{A.7}$$

we obtain the following three equations:

$$\left(-\lambda^2 - \frac{1-\nu}{2}n^2 + \Delta\right)A + \left(\frac{1+\nu}{2}\lambda n - \lambda n\, n_\varphi\right)B + \left(-\nu\lambda + n_\varphi \lambda\right)C = 0.$$

$$\begin{aligned} \frac{1+\nu}{2}\lambda n\, A + \left\{ -\frac{1-\nu}{2}\lambda^2 - n^2 - \beta\left[n^2 + (1-\nu)\lambda^2\right] - n_x\lambda^2 + \Delta \right\} B \\ + \left(n + \beta n^3 + \beta\lambda^2 n\right) C = 0. \end{aligned} \tag{A.8}$$

$$\begin{aligned} -\nu\lambda A + \left\{ n + \beta\left[(2-\nu)\lambda^2 n + n^3\right] + n_\varphi n \right\} B \\ + \left[-1 - \beta(\lambda^4 + 2\lambda^2 n^2 + n^4) - n_x\lambda^2 - n_\varphi n^2 + \Delta \right] C = 0 \end{aligned}$$

For a nontrivial solution the determinant of the coefficients of A, B, C in Eqs. (A.8) must vanish, leading to the frequency equation:

$$\Delta^3 - K_2'\Delta^2 + K_1'\Delta - K_0' = 0 \tag{A.9}$$

where, reverting to the notations $\bar{n}_\varphi$, $\bar{n}_x$,

$$\begin{aligned} K_0' &= K_0 + a_1\bar{n}_\varphi + a_2\bar{n}_x + a_3\bar{n}_x\bar{n}_\varphi + a_4\bar{n}_x^2 + a_5\bar{n}_\varphi^2 \\ K_1' &= K_1 + b_1\bar{n}_\varphi + b_3\bar{n}_x + n^2\lambda^2\bar{n}_x\bar{n}_\varphi + \lambda^4\bar{n}_x^2 \\ K_2' &= K_2 + n^2\bar{n}_\varphi + 2\lambda^2\bar{n}_x \end{aligned} \tag{A.10}$$

When the internal pressure vanishes, the coefficients K_0', K_1', K_2' reduce to K_0, K_1, K_2 which are given by Arnold and Warburton*, (Ref. 1). The expressions are:

*When $\bar{n}_x = \bar{n}_\varphi = 0$, Eq. (A.8) differ from Ref. 1 in the last term in the second equation; instead of $\beta\lambda^2 n C$, Ref. 1 reads $(2-\nu)\beta\lambda^2 n C$. The effect on numerical results due to this difference is negligible.

$$K_0 = \frac{1}{2}(1-\nu)^2(1+\nu)\lambda^4 + \frac{1}{2}(1-\nu)\beta\Big[(\lambda^2+n^2)^4 - 2(4-\nu^2)\lambda^4 n^2 -$$

$$- 8\lambda^2 n^4 - 2n^6 + 4(1-\nu^2)\lambda^4 + 4\lambda^2 n^2 + n^4\Big]$$

$$K_1 = \frac{1}{2}(1-\nu)(\lambda^2+n^2)^2 + \frac{1}{2}(3-\nu-2\nu^2)\lambda^2 + \frac{1}{2}(1-\nu)n^2$$

$$+\beta\Big[\frac{1}{2}(3-\nu)(\lambda^2+n^2)^3 + 2(1-\nu)\lambda^4 - (2-\nu^2)\lambda^2 n^2 - \frac{1}{2}(3+\nu)n^4 + 2(1-\nu)\lambda^2 + n^2\Big]$$

$$K_2 = 1 + \frac{1}{2}(3-\nu)(\lambda^2+n^2) + \beta\Big[(\lambda^2+n^2)^2 + 2(1-\nu)\lambda^2 + n^2\Big].$$

$$a_1 = \frac{1-\nu}{2}n^2(n^2-\lambda^2)^2 - \frac{1-\nu}{2}n^4 + \frac{\nu(1-\nu)}{2}\lambda^4 - \frac{(2-3\nu-\nu^2)}{2}\lambda^2 n^2 - \nu\lambda n$$

$$-\beta\left\{(\lambda^2+n^2)\left(n^2\lambda^2 + \frac{1-\nu}{2}n^4 + \nu\lambda n\right) + \frac{1+\nu}{2}\lambda^2 n^2\Big[(2-\nu)\lambda^2 + n^2 - (\lambda^2+n^2)^2\Big]\right\}$$

$$a_2 = \lambda^2\left\{(1-\nu^2)\lambda^2 + \frac{1-\nu}{2}\Big[n^2 + (n^2-\lambda^2)^2\Big] + \beta(\lambda^2+n^2)^2\left(\lambda^2 + \frac{1-\nu}{2}n^2\right)\right\}$$

$$a_3 = \lambda^2\Big[\frac{3-\nu}{2}\lambda^2 n^2 + \frac{1-\nu}{2}n^4 + \nu\lambda^2\Big]$$

$$a_4 = \lambda^4\left(\lambda^2 + \frac{1-\nu}{2}n^2\right)$$

$$a_5 = \frac{1+\nu}{2}\lambda^2 n^2(n^2-1)$$

$$b_1 = \frac{3-\nu}{2}n^4 + 2\lambda^2 n^2 - n^2 + \nu\lambda^2 - \beta n^2(\lambda^2+n^2)$$

$$b_2 = \frac{5-\nu}{2}\lambda^4 + \frac{5-2\nu}{2}\lambda^2 n^2 + \lambda^2 + \beta\lambda^2(\lambda^2+n^2)^2.$$

APPENDIX B

Simplification of the Frequency Equation

Under various additional assumptions the frequency equation can be simplified.

Reissner (10) introduced the assumption that in "predominantly flexural" vibrations of the cylinder the tangential inertia force can be neglected. Under this assumption the last terms in the first two equations of (A.3) and the two Δ terms in the first two equations of (A.8) are removed. The frequency equation then becomes a linear equation

$$\bar{K}_1' \Delta - K_0' = 0$$

Where $\bar{K}_1'$ is somewhat different from K_1'. This equation, however, is not identical with Reissner's equation (1) or (7). In order to obtain the latter, it is necessary to consider sufficiently large values of n, so that the shell panel between the nodal lines can be approximated by a slightly curved plate.

In the unpressurized case, Arnold and Warburton (1) give a simplified linear expression for the frequency parameter Δ that is valid for sufficiently small values of n.

Stern's (7) approximation consists in neglecting all terms in Eq. (A.5) except those involving w within the square bracket in the last equation. The frequency equation is therefore

$$\Delta = \lambda^2 n_x + n^2 n_\varphi$$

The justification of this approximation is not apparent. Comparison with Eq. 7 shows that, numerically, it is a fair approximation for small values of λ and moderate values of n, provided that β is small.

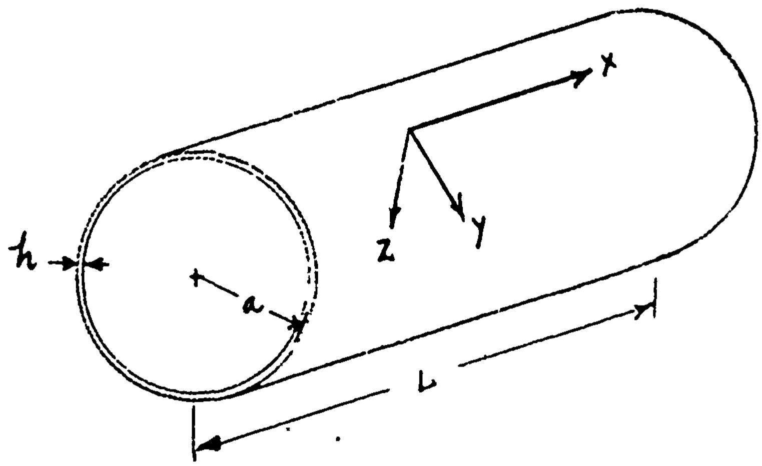


Fig 1a

Coordinate system

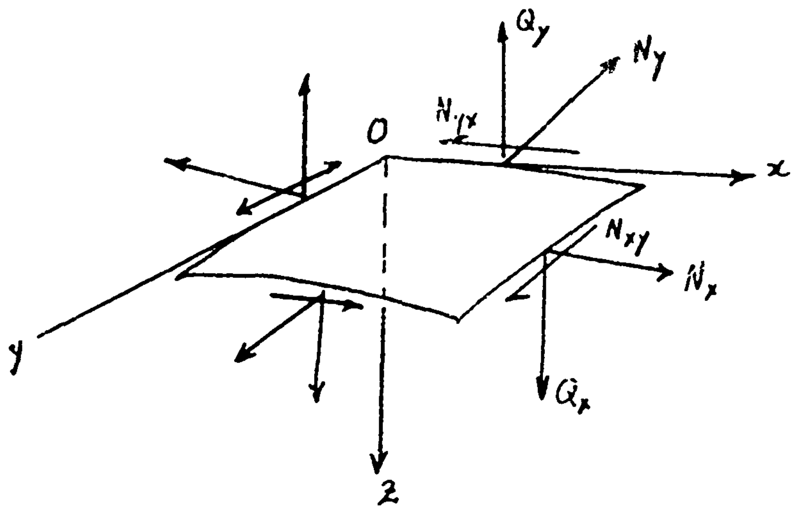


Fig 1b

Stress resultants

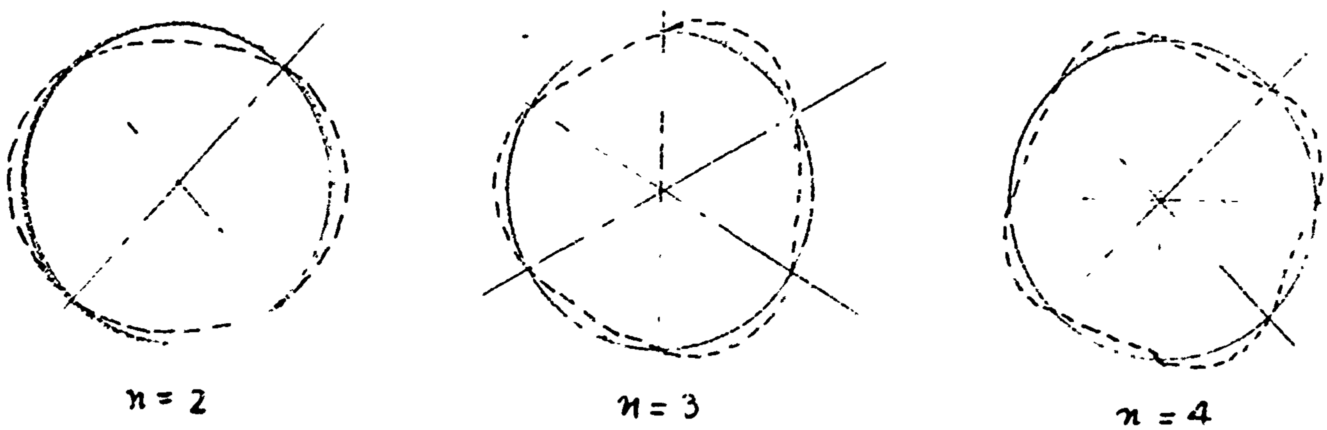


Fig 2 Circumferential waves

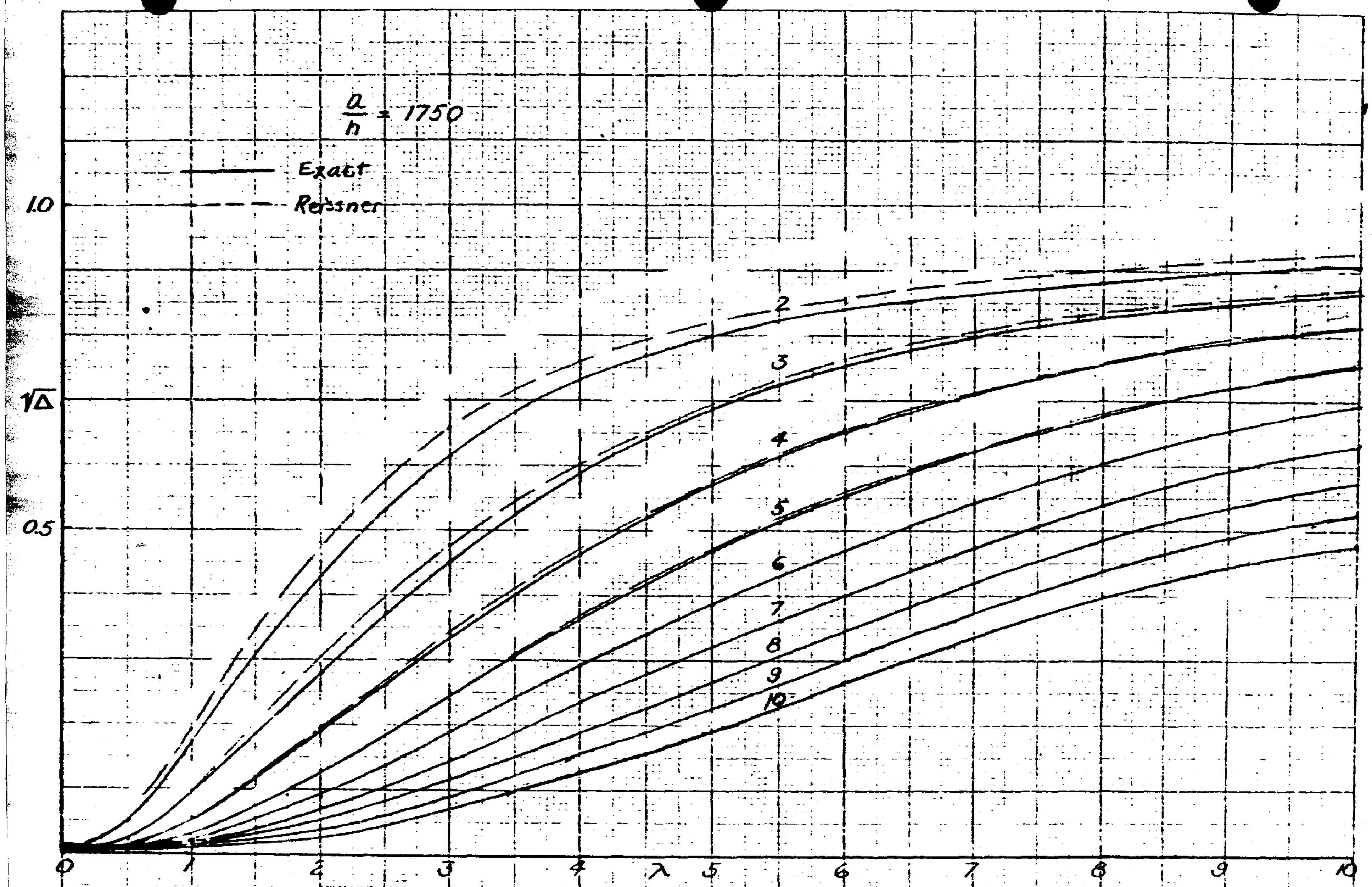


FIG. 3a Frequency parameter $\sqrt{\Delta}$ vs axial wave length parameter λ for various circumferential wave no. n

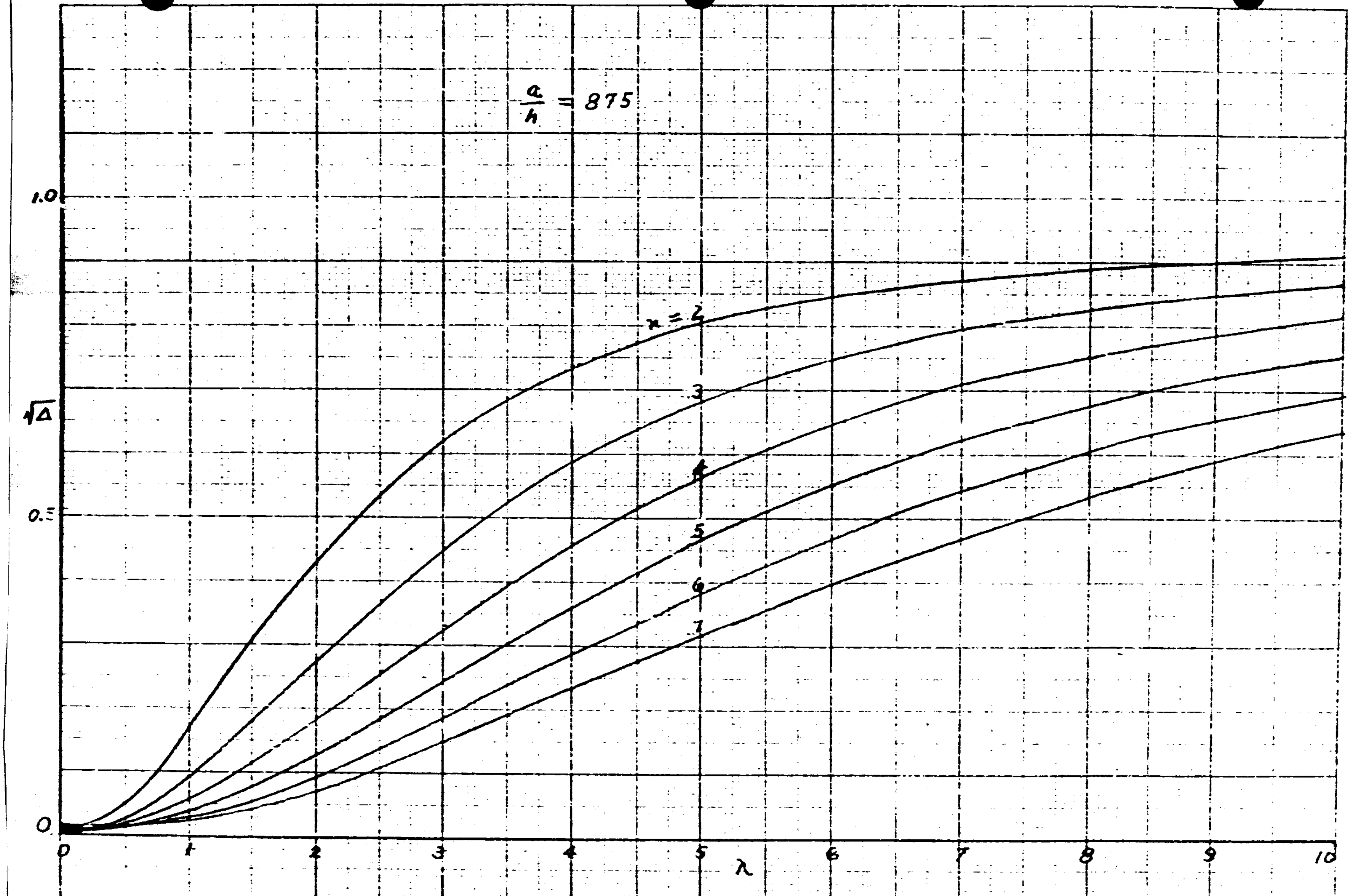


FIG 3b Frequency parameter $\sqrt{\Delta}$ vs axial wave length parameter λ for various circumferential wave number n.

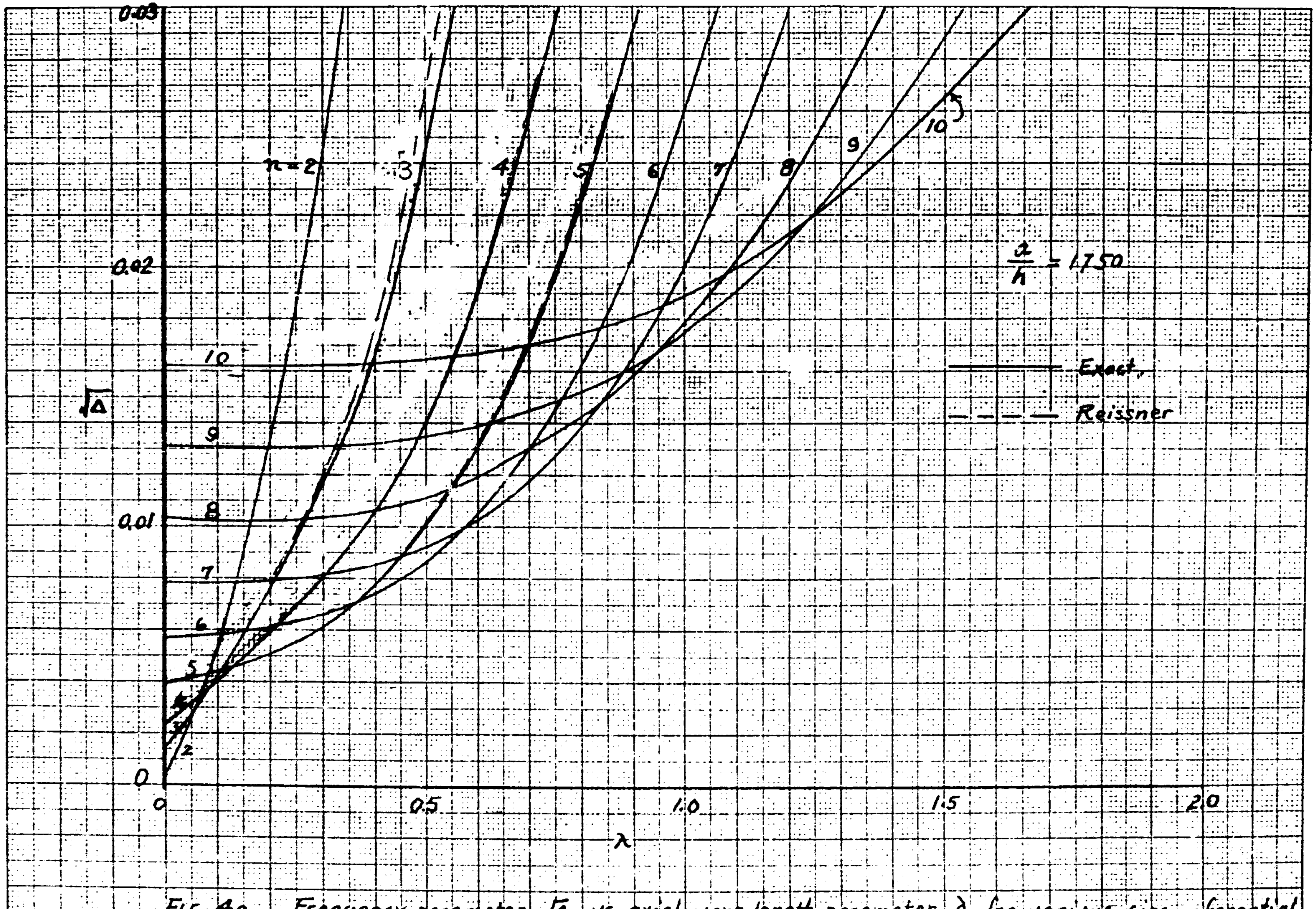


FIG. 4a. Frequency parameter $\sqrt{\Delta}$ vs axial wave length parameter λ for various circumferential wave number n.

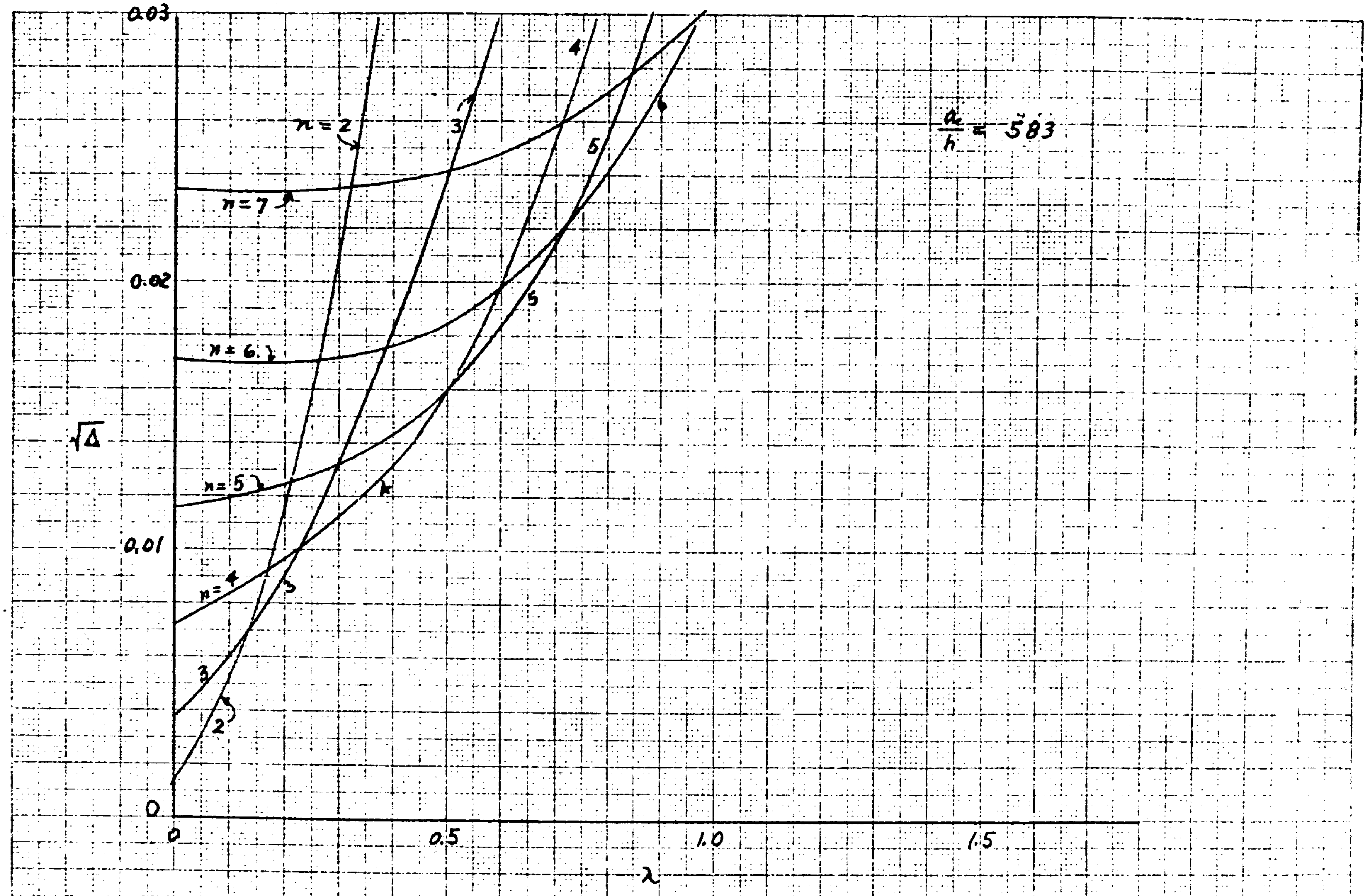


FIG 4b. Frequency parameter $\sqrt{\Delta}$ vs. axial wave length parameter λ for various circumferential wave number n.

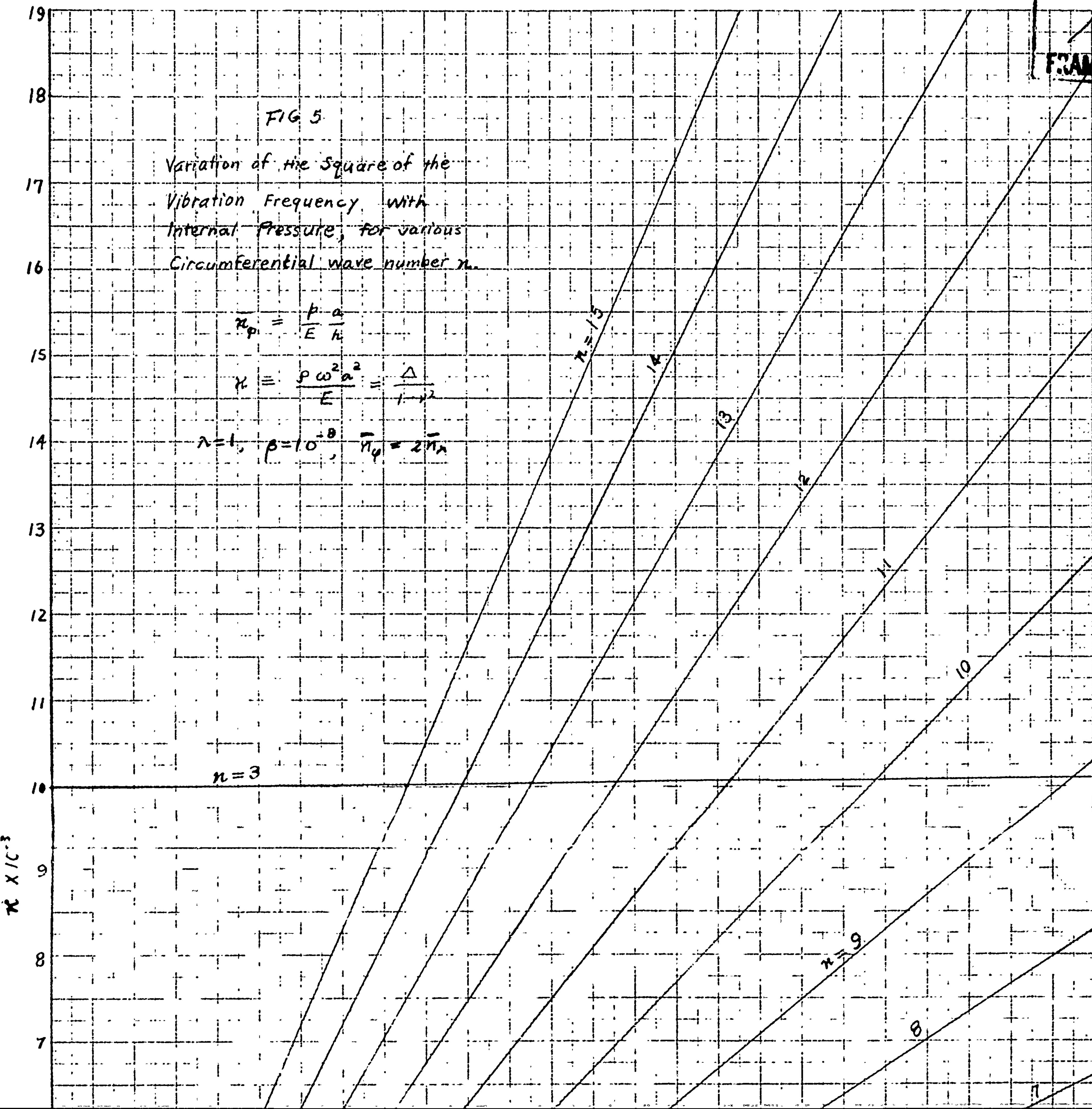


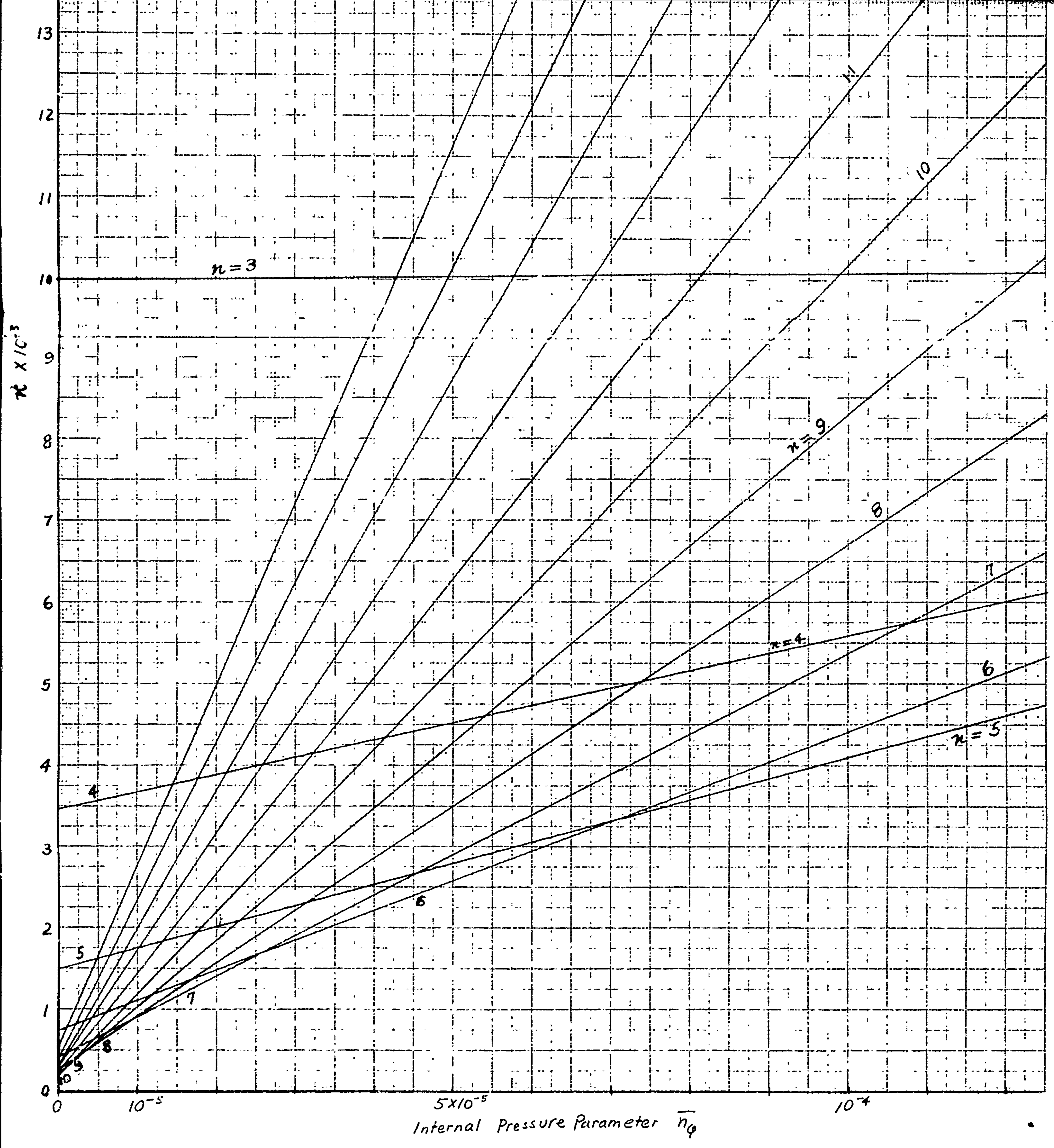
FIG 5

Variation of the Square of the Vibration Frequency with Internal Pressure, for various Circumferential wave number n.

$$\bar{\pi}_{\varphi} = \frac{p}{E}\frac{a}{h}$$

$$\varkappa = \frac{\rho \omega^2 a^2}{E} = \frac{\Delta}{1-\nu^2}$$

$\lambda = 1$, $\beta = 10^{-8}$, $\bar{\pi}_{\varphi} = 2\bar{\pi}_{x}$



13
12
11
10
9
8
7
6
5
4
3
2
1
0
$\kappa \times 10^{-3}$
$n=3$
11
10
$n=9$
8
7
$n=4$
6
$n=5$
4
5
6
7
8
9
10
0
10^{-5}
5×10^{-5}
10^{-4}
Internal Pressure Parameter $\overline{n_\varphi}$

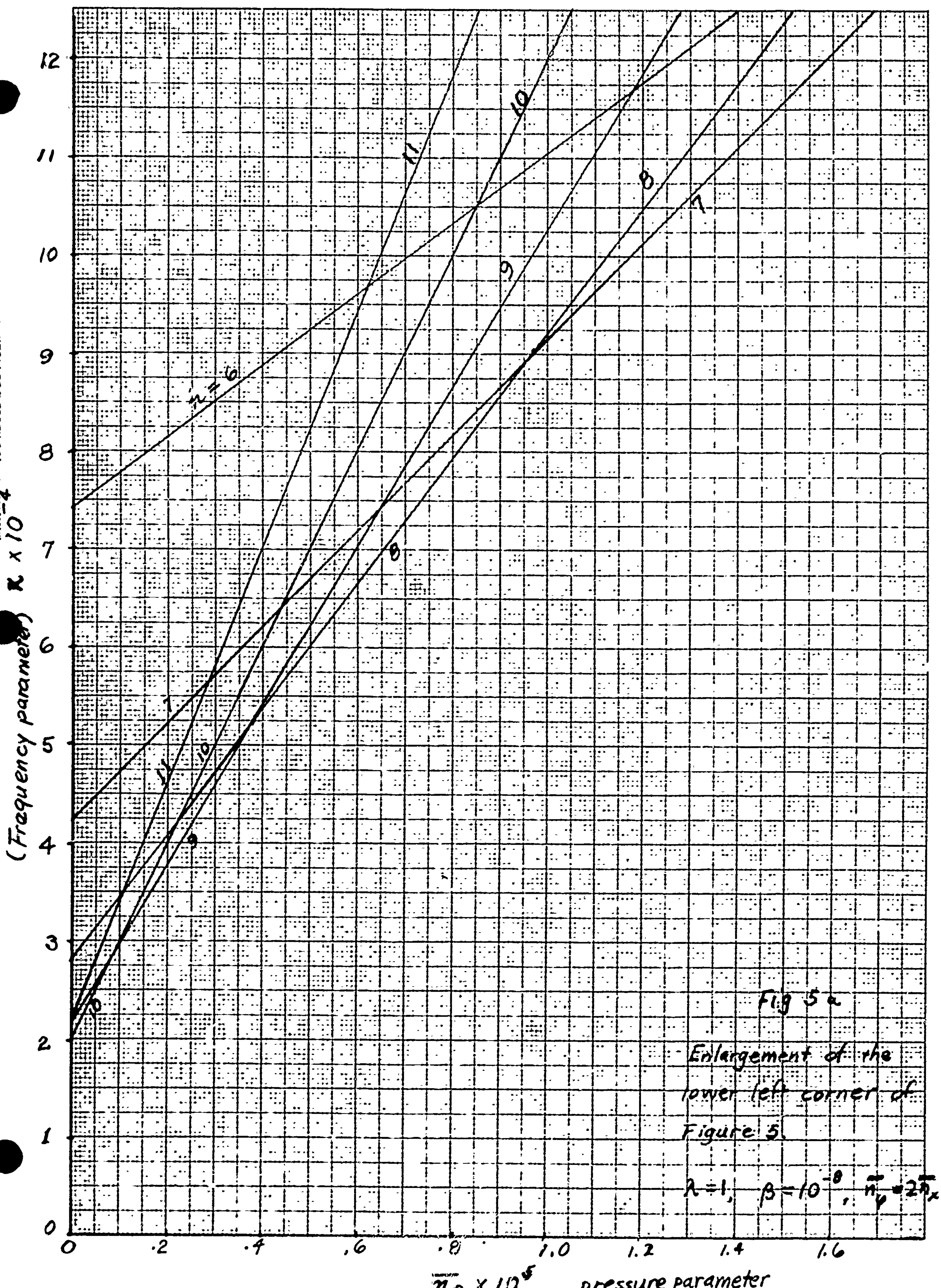


Fig 5a

Enlargement of the lower left corner of Figure 5.

$\lambda = 1, \quad \beta = 10^{-8}, \quad \bar{n}_y = 2\bar{n}_x$

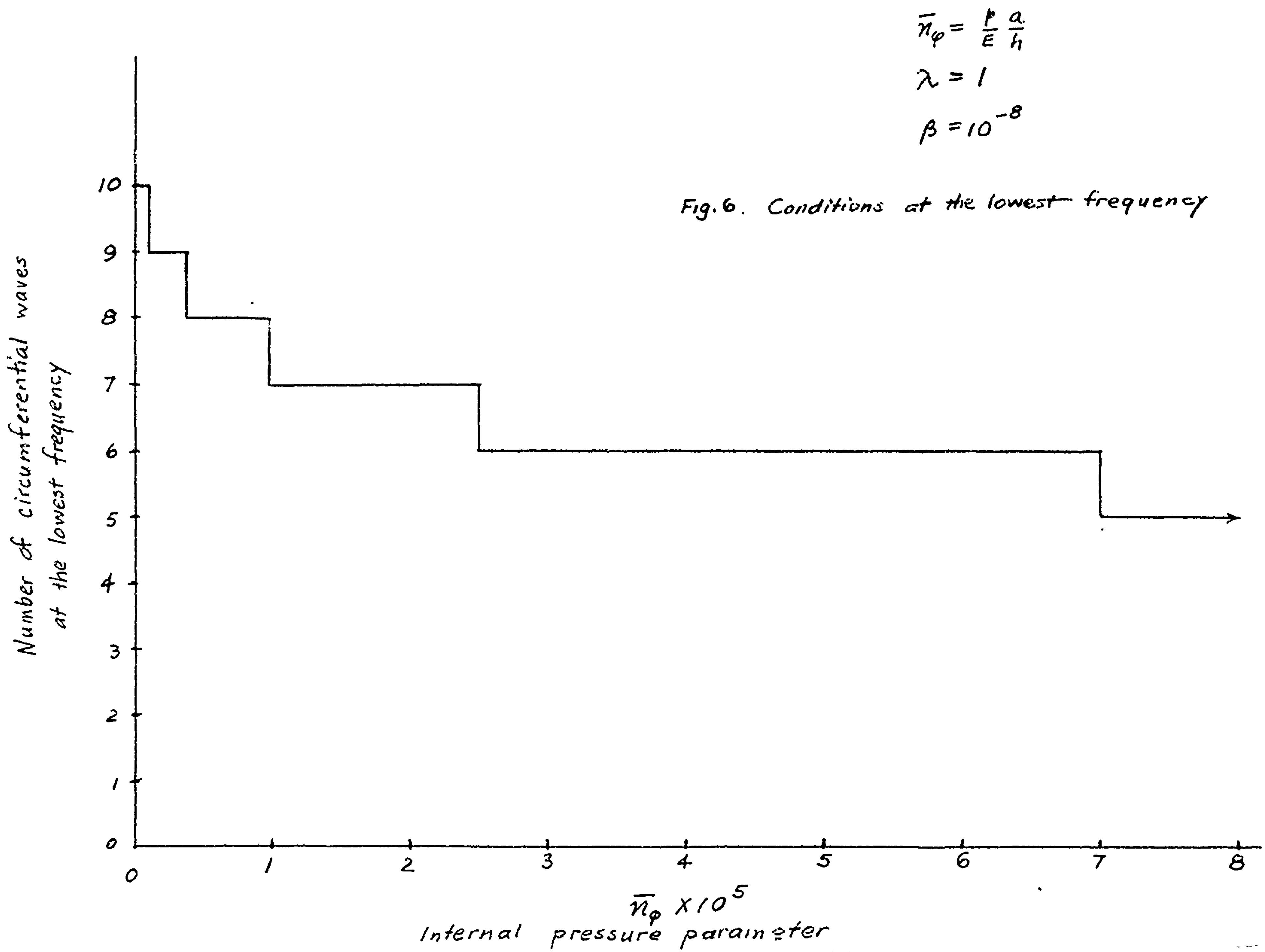


Fig. 6. Conditions at the lowest frequency

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